Please cite this paper as follows:

Hong-Ki Hong and Hwa-Shang Lan, An Exact Constitutive Solution for Thermal-Elastic-Plastic Materials, in J.-P. Boehler and A.S. Khan (eds.) Anisotropy and Localization of Plastic Deformation, Proceedings of Plasticity '91: the Third International Symposium on Plasticity and Its Current Applications pp.548-551, 1991.

AN EXACT CONSTITUTIVE SOLUTION FOR THERMAL-ELASTIC-PLASTIC MATERIALS

HONG-KI HONG* and HWA-SHANG LAN

Department of Civil Engineering,
 National Taiwan University, Taipei 10764, Taiwan, R.O.C.

Chung-Shan Institute of Science and Technology,
 P.O. BOX 90008-15-13, Lungtan, Taoyuan 32526, Taiwan, R.O.C.

ABSTRACT

An integral constitutive equation is proposed for thermal-elastic-plastic materials with the yield condition of the form $f(J_2, \bar{\epsilon}^p, \theta) = 0$, where stress is completely characterized by strain and temperature histories and which is applicable to mixed, isotropic, and kinematic hardening. When the von Mises type yield condition $J_2 - \kappa^2(\theta) = 0$ is specified and a constant strain rate within a time step is assumed, an exact constitutive solution can be obtained therefrom. For illustration a numerical experiment to simulate the tensile test of a uniaxial coupon is conducted.

INTRODUCTION

The constitutive descriptions for thermal-elastic-plastic materials are often conclusively characterized by a rate-form constitutive equation [1,2] and therefrom considerable numerical integration algorithm has been developed. The aim of this paper is to derive an integral representation of the conventional rate constitutive descriptions for thermal-elastic-plastic materials with the yield condition of the form $f(J_2, \bar{\epsilon}^p, \theta) = 0$, where the arguments are the second invariant of the deviatoric stress ξ_{ij} , effective plastic strain, and temperature change respectively and based on this frame to propose a new integration algorithm. The motivation is that for a body deformed with memory, the stress may be expressed as a functional of deformation and temperature [3]. It is seen that the proposed stress expression is exactly represented by an integral which is a function of the history of the flow variables, strain and temperature. An exact constitutive solution is obtained when the von Mises-type yield condition $J_2 - \kappa^2(\theta) = 0$ is specified and a constant rate within a time step is assumed. Also, under any non-linear strain hardening, the algorithm is applicable without any difficulty. For illustration a numerical experiment to simulate the uniaxial tensile test of a axial-symmetric specimen is examined.

FORMULATIONS

At all stages of the loading history, the total deformation rate tensor is considered to

be decomposed into a reversible part, d_{ij}^{rv} , and an inelastic (plastic) part, d_{ij}^p :

$$d_{ij} = d_{ij}^{rv} + d_{ij}^p \quad (1)$$

and the rate of stress tensor $\dot{\sigma}_{ij}$ is expressed as (van der Lugt & Huetink [1])

$$\dot{\sigma}_{ij} = E_{ijkl} d_{ij}^{rv} + L_{ij} \dot{\theta} \quad (2)$$

where $E_{ijkl} = \frac{E}{2(1+\nu)} (\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} + \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{kl})$, $L_{ij} = -\beta\delta_{ij}$, $\beta = \frac{E\alpha}{1-2\nu}$, and $\theta = T - T_0$, with E as Young's modulus, ν Poisson's ratio, β the thermal modulus, α the coefficient of thermal expansion, T the absolute temperature, and T_0 the initial temperature. The elastic moduli E and ν are assumed to be insensitive to small temperature change θ [2]. If the von Mises-type yield condition is applied,

$$f(J_2, \bar{e}^p, \theta) = \frac{1}{2}\xi_{ij}\xi_{ij} - \frac{1}{3}\kappa^2(\bar{e}^p, \theta) = 0, \quad \text{where } \xi_{ij} = s_{ij} - \alpha_{ij} \quad (3)$$

in which s_{ij} is the stress deviator, α_{ij} deviatoric kinematic hardening tensor (back stress), and κ yield stress function of material depending on the effective plastic strain $\bar{e}^p (\equiv \int_0^t (2/3 d_{ij}^p(\tau) d_{ij}^p(\tau))^{1/2} d\tau)$ and θ . The associative flow rule and kinematic hardening law are respectively

$$d_{ij}^p = \begin{cases} \frac{3\bar{e}^p}{2\kappa}\xi_{ij} & \text{iff } f = \dot{f} = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and } \dot{\alpha}_{ij} = \frac{2}{3}h_{,\bar{e}^p}(\bar{e}^p, \theta)d_{ij}^p \quad (\text{Prager-Ziegler type}) \quad (4)$$

where $h_{,\bar{e}^p}$, abbreviation for $\partial h / \partial \bar{e}^p$, depends on $\bar{e}^p$ and θ .

Based on eqns (1) through (4), and after performing appropriate manipulations, eqn (2) can be decomposed into a volumetric and a deviatoric part [4]:

• for the volumetric part:

$$\frac{1}{3}\text{trace}(\sigma) = K \int_0^t \text{trace}(\dot{d}(\tau)) d\tau - \beta\theta \quad (5)$$

• for the deviatoric part:

$$\xi_{ij}(t) = e^{-\Phi(t)} \left(\xi_{ij}(0) + 2G \int_0^t e^{\Phi(\tau)} d'_{ij}(\tau) d\tau \right) \quad (6)$$

where d'_{ij} is the deviator of d_{ij} and the plastic parameter Φ ,

$$\Phi(t) = \int_0^t \frac{(3G + h_{,\bar{e}^p}(\bar{e}^p, \theta))\bar{e}^p}{\kappa(\bar{e}^p, \theta)} d\tau \quad (7)$$

is a monotonically increasing function of the amount of plastic flow.

For purpose of numerical implementation we rewrite eqn (6) in the tensor form

$$\xi = \frac{1}{A}(\xi_n + 2G \int_{t_n}^t A(\tau) d'(\tau) d\tau), \quad t \geq t_n, \quad \text{where } A(t) = e^{\Phi(t) - \Phi(t_n)} \quad (8)$$

where $(\cdot)_n$ denotes the converged solutions $(\cdot)$ at time $t = t_n$. To enforce the consistency condition at the end of each time step, we substitute eqn (8) into (3) and obtain

$$\left(\frac{\kappa A}{\kappa_n} \right)^2 = 1 + \omega \xi_n : \int_{t_n}^t A d' d\tau + \eta \int_{t_n}^t A d' d\tau : \int_{t_n}^t A d' d\tau \quad (9)$$

where $\omega = 6G/\kappa_n^2$ and $\eta = 6G^2/\kappa_n^2$. This equation closely relates the effective plastic strain $\bar{\epsilon}^p$ with the strain rate deviator $\mathbf{d}'$ and temperature θ .

For Constant Strain Rate

Under the consideration of constant strain rate, eqns (8) and (9) can be respectively expressed as

$$\xi = \frac{1}{A}(\xi_n + 2GR\mathbf{d}') \quad (10)$$

and

$$\frac{\kappa A}{\kappa_n} = F(R)^{1/2}, \text{ where } F(R) \equiv 1 + \bar{\omega}R + \bar{\eta}R^2 \quad (11)$$

where $R = \int_{t_n}^t A(\tau) d\tau$, $\bar{\omega} = \omega \xi_n : \mathbf{d}'$, and $\bar{\eta} = \eta \mathbf{d}' : \mathbf{d}'$

Exact Solution

Consider that κ and $h_{,\bar{\epsilon}^p}$ are independent on $\bar{\epsilon}^p$, i.e., $\kappa = Y(\theta)$ and $h_{,\bar{\epsilon}^p} = D(\theta)$. The exact solution of eqn (11) is

$$R(t) = \frac{1}{\sqrt{\bar{\eta}}} \sinh(\sqrt{\bar{\eta}}Z(t)) + \frac{\bar{\omega}}{2\bar{\eta}} (\cosh(\sqrt{\bar{\eta}}Z(t)) - 1), \text{ where } Z(t) = \int_{t_n}^t \frac{Y_n}{Y(\theta(\tau))} d\tau \quad (12)$$

and whereupon

$$A(t) = \dot{R}(t) = \frac{Y_n}{Y(\theta)} \left(\cosh(\sqrt{\bar{\eta}}Z(t)) + \frac{\bar{\omega}}{2\sqrt{\bar{\eta}}} \sinh(\sqrt{\bar{\eta}}Z(t)) \right) \quad (13)$$

ξ can be found from eqn (10). The $\dot{\bar{\epsilon}}^p$ and therefore α can be easily obtained with the help of eqns (4), (7), and (10).

Non-linear hardening laws

In case that κ and h are arbitrary functions of $\bar{\epsilon}^p$ and θ , the present constitutive model problem is equivalently reduced to the initial value problems given by

$$\dot{\bar{\epsilon}}^p = \hat{f}(t, \bar{\epsilon}^p, R) \equiv \frac{\kappa_n}{3G + h_{,\bar{\epsilon}^p} + \kappa_{,\bar{\epsilon}^p}} \left(\frac{\bar{\omega} + 2\bar{\eta}R}{2F(R)^{1/2}} - \frac{\kappa_{,\theta} \dot{\theta}}{\kappa_n} \right), \quad \bar{\epsilon}^p(t_n) = \bar{\epsilon}_n^p \quad (14)$$

$$\dot{R} = \hat{g}(t, \bar{\epsilon}^p, R) \equiv \frac{\kappa_n}{\kappa} F(R)^{1/2}, \quad R(t_n) = 0 \quad (15)$$

in which the dependence on time t of functions $\hat{f}$ and $\hat{g}$ results from the dependence on temperature variation $\theta(t)$ of κ and h .

ILLUSTRATION

We use the present integration algorithm to calculate the temperature variation of an axial-symmetric specimen which is tensioned by a constant velocity 0.01mm/sec, see Figure 1. Consider the energy equation governed by

$$\rho C_v \dot{\theta} - \lambda \nabla^2 \theta = \sigma : \mathbf{d}^p - \beta \text{trace}(\mathbf{d}^{rv})(T_0 + \theta) \quad (16)$$

where ρ is the density, C_v specific heat at constant strain, and λ coefficient of heat conductivity. Both terms on the r.h.s. of eqn (16) represent the heat generation due to the mechanical work. The results are shown in figures 2 and 3.

Figure 1. Tensile specimen and its material properties

Figure 2. The predicted stress of the uniaxial test

Figure 3. The predicted temperature change of the uniaxial test

CONCLUSION

A new stress representation for the thermal-elastic-plastic materials is proposed. In contrast to the use of a stress rate, the stress is expressed as a functional of deformation and temperature through the existence of a kernel function $\exp(\Phi(\tau) - \Phi(t))$ with $\tau \leq t$. This is meaningful in both theoretical and experimental approach and deserving of further research.

REFERENCES

1. van der Lugt, J., and Huetink, J., Thermal mechanically coupled finite element analysis in metal-forming process. Comp. Meth. Appl. Mech. Eng., 1986, **54**, 145-160.
2. Argyris, J. H., Doltsinis, J. St., Pimenta, P. M., and Wustenberg, H., Thermomechanical response of solids at high strain-natural approach, Comp. Meth. Appl. Mech. Eng., 1982, **32**, 3-57.
3. Malvern, L. E., Introduction to the Mechanics of A Continuous medium, Prentice-Hall Inc., 1969., pp. 319-324.
4. Hong, H.-K., and Lan, H.-S., A new point-of-view at plastic materials of the prandtl-reuss type, In Advance in Plasticity 1989, ed. A.S. Khan and M. Tokuda, 1989, pp. 95-98.